ON INTERNAL SYNCHRONIZATION OF ALMOST LIKE DYNAMIC OBJECTS UNDER THE ACTION OF WEAK LINEAR CONSTRAINTS

(O VNUTRENNEI SINKHRONIZATSII POCHTI ODINAKOVYKH DINAMICHESKIKH OB"EKTOV POD DEISTVIEM SLABYKH LINEINYKH SVIAZEI)

PMM Vol.28, № 2, 1964, pp.216-220

R.F. NAGAEV

(Leningrad)

(Received October 24, 1963)

We consider the general case of almost identical objects synchronized under the action of weak linear couplings. Conditions are obtained for the existence and stability of synchronous motions, as well as the equations for determination of "generated phases".

In almost all fields of science and technology one has to consider systems of several correlated dynamic objects functioning together*. One speaks of correlated or synchronous motions of objects in a system if the fundamental parameters which characterize the "rhythm" of motion of these objects coincide. For periodic motions one is concerned with the coincidence of their periods. The synchronization of objects is realized by means of couplings between them, taken in the broad sense of this term.

In the following, just as in [1], we shall single out the case of "strong" couplings, by means of which the object may be subjected to a considerable influence. The case of such synchronization is trivial in the deterministic sense and, in particular, yields the fact that the value of the motion of one of the objects necessarily determines the motion of all other objects.

Of fundamental interest is the synchronization in the presence of weak couplings, whose effects on the objects are small and which only slightly distort the character of the motion of the objects. In view of this it is understandable that a system with weak couplings may be made synchronous, generally speaking, only in the case when the initially isolated objects, not coupled to each other, execute motions which are close, at least during a finite and sufficiently large interval of time, to the sought synchronous one. In other terms, the quantities which characterize the degree of non-synchronous behavior of the system of initially isolated objects, should be of the same order of magnitude as the parameter which reflects the "strength" of the couplings. It is understandable that this condition is satisfied, above all, by systems of almost identical objects.

---

* Numerous examples of such systems, as well as a large bibliography, are given in the paper of Blekhman [1].

It is necessary to point out that in a system of isolated autonomous objects an infinity of motions is possible, which are close to the synchronous ones. Indeed, almost synchronous motions, realizable in a system of isolated objects, may be displaced one with respect to the other in phase, by virtue of their autonomous behavior. This multitude of motions represents a family which depends at least on m arbitrary constants, where m is the total number of objects. If one does not consider the slight nonsynchronous motions of isolated objects, then the question arises concerning the selection from this family of such a motion*, which qualitatively and quantitatively is close to the sought synchronous regime in the interconnected system. This, indeed, represents the essence of the problem of self-phasing, which always accompanies the fundamental problem on the determination of the regions of existence and stability of synchronous regimes of motion of systems of dynamical objects subjected to weak couplings.

1. In the following we shall assume that the motion of an interconnected system of objects is described by ordinary differential equations with a small parameter. We shall use, thereby, the theorem on the existence and stability of solutions of a definite form, which is a generalization of a corresponding theorem discussed in the monograph by Malkin [2]. We shall omit the proof of this theorem, because it can be carried out in an analogous manner, and we shall limit ourselves to its formulation.

Let us assume that we have an autonomous system

$$\frac{dx_s}{dt} = X_s(x_1, \ldots, x_n) + \mu f_s(x_1, \ldots, x_n, \mu) \qquad (s = 1, \ldots, n) \tag{1.1}$$

where the functions X_s and f_s are analytic with respect to the variables $x_1, \ldots, x_n$ and a small parameter μ in some region G. Further, we shall assume that the functions X_s and f_s are pèriodic in 2π with respect to the variables $x_1, \ldots, x_l$ $(l \leqslant n)$.

The generating system

$$\frac{dx_s^\circ}{dt} = X_s(x_1^\circ, \ldots, x_n^\circ) \qquad (s = 1, \ldots, n) \tag{1.2}$$

admits a family of solutions of the type

$$x_s^\circ = \varphi_s(t, h_1, \ldots, h_k) \tag{1.3}$$

which depends, aside from a constant h which can be added to t, on k arbitrary constants $h_1, \ldots, h_k$. Thereby let

$$\begin{aligned} \varphi_s(t+T) &= 2\pi + \varphi_s(t) \qquad (s = 1, \ldots, l) \\ \varphi_s(t+T) &= \varphi_s(t) \qquad (s = l+1, \ldots, n) \end{aligned} \tag{1.4}$$

where the period T does not depend on the constants $h_1, \ldots, h_k$.

Under the assumptions adopted, τ-periodic and linearly increasing solutions of system (1.1), becoming for $\mu = 0$ the solutions of the generated

* In problems of synchronization, the motions, which are possible in each object in the absence of couplings, are assumed to be known; generally their representation does not present any difficulties.

system (1.2) of the form $\varphi_s(t, h_1^*, \ldots, h_k^*)$, are possible if the constants $h_1^*, \ldots, h_k^*$ satisfy the system

$$P_i(h_1^*, \ldots, h_k^*) = \sum_{\beta=0}^{n} \int_0^T f_\beta\, [\varphi_1(t, h_1^*, \ldots, h_k^*), \ldots, \varphi_n(t, h_1^*, \ldots h_k^*), 0]\, \psi_{\beta i}(t)\, dt = 0 \qquad (i = 1, \ldots, k) \tag{1.5}$$

Here the functions $\psi_{\beta i}(t)$ $(i = 1, \ldots, k)$ are T-periodic solutions of the linear system associated with the equations in variations of system (1.2)

$$\frac{dz_s}{dt} + p_{1s}z_1 + \ldots + p_{ns}z_n = 0 \qquad (s = 1, \ldots, n)$$

$$p_{sj} = \left(\frac{\partial X_s}{\partial x_j}\right)_{x_r = \varphi_r(t, h_1^*, \ldots, h_k^*)} \tag{1.6}$$

Further, the functions $\psi_{\beta i}(t)$ must satisfy conditions

$$\sum_{\alpha=1}^{n} \frac{\partial \varphi_\alpha(t, h_1^*, \ldots, h_k^*)}{\partial h_j^*} \psi_{\alpha i}(t) = \begin{cases} 0 & (i \neq j) \\ 1 & (i = j) \end{cases}$$

$$\sum_{\alpha=1}^{n} \varphi_\alpha^{\cdot}(t, h_1^*, \ldots, h_k^*) \psi_{\alpha i}(t) = \begin{cases} 0 & (i \neq k+1) \\ 1 & (i = k+1) \end{cases} \qquad \begin{pmatrix} i = 1, \ldots, k+1 \\ j = 1, \ldots, k+1 \end{pmatrix} \tag{1.7}$$

The first approximation to the true period of motion is calculated by Formula

$$\tau = T + \mu\alpha^* \tag{1.8}$$

$$\alpha^* = -\sum_{\beta=1}^{n} \int_0^T f_\beta [\varphi_1(t, h_1^*, \ldots, h_k^*), \ldots, \varphi_n(t, h_1^*, \ldots, h_k^*)]\, \psi_{\beta k+1}(t)\, dt$$

For the stability of such a solution of system (1.1) it is sufficient that all roots of Equation

$$\begin{vmatrix} \partial P_1/\partial h_1^* - \varkappa & \ldots & \partial P_1/\partial h_k^* \\ \cdot \; \cdot \; \cdot & \cdot \; \cdot \; \cdot & \cdot \; \cdot \; \cdot \\ \partial P_k/\partial h_1^* & \ldots & \partial P_k/\partial h_k^* - \varkappa \end{vmatrix} = 0 \tag{1.9}$$

satisfy the conditions $\operatorname{Re} \varkappa < 0$.

2. Let us consider the internal synchronization of a system of almost-like dynamic objects in the presence of weak linear couplings. The term internal synchronization is commonly used in the case when no external effects are imparted to the objects by means of such couplings.

Let the motion m of almost-like objects in a coupled system be described by differential Equations

$$x_{si}^{\cdot} = X_s(x_{1i}, \ldots, x_{ni}) + \mu\Big(f_{si}(x_{1i}, \ldots, x_{ni}) + \sum_{j=1}^{k} V_{si}^{(j)}(x_{1i}, \ldots, x_{ni})\, u_j\Big) \tag{2.1}$$

Here and in the following a dot denotes differentiation with respect to time.

The weak linear couplings will be characterized by the system

$$u_j^{\cdot} = a_{1j}u_1 + \ldots + a_{kj}u_k + \sum_{r=1}^{m} U_{rj}(x_{1r}, \ldots, x_{nr}) \qquad (j = 1, \ldots, k) \quad (2.2)$$

In equations (2.1) and (2.2) the quantities a_{ij} $(i, j = 1, \ldots, k)$ will be constant, and the functions X_s, f_{si}, $V_{si}^{(j)}$ and U_{rj} are such that the theorem formulated in the preceding section becomes applicable. The function f_{si} characterizes the degree of difference of objects, and the functions $V_{si}^{(j)}$ and U_{rj} characterize the coupling between the objects.

In a system of identical isolated objects, which is considered in the generating approximation

$$\overset{\circ}{x}_{si}^{\cdot} = X_s(\overset{\circ}{x}_{1i}, \ldots, \overset{\circ}{x}_{ni}) \qquad (s = 1, \ldots, n, \ i = 1, \ldots, m) \quad (2.3)$$

the synchronous motions

$$\overset{\circ}{x}_{si} = \varphi_s(t + \alpha_i) \quad (2.4)$$

are realizable for all objects; they are all alike and completely determined within an arbitrary phase shift.

The functions $\varphi_s(t)$ are such that the condition is satisfied in any case

$$\varphi_s^{\cdot}(t + T) = \varphi_s^{\cdot}(t)$$

Thus, the period of synchronous motions T does not depend on the values of the arbitrary phases α_i . Further, by virtue of the autonomous behavior of the interconnected system, one of the phases, for example α_m, may be taken as being equal to zero. Let us assume that all roots of the characteristic equation

$$\| a_{ij} - \delta_{ij}\rho \| = 0 \qquad (\delta_{ij} = 0 \ \text{при} \ i \neq j, \ \ \delta_{ij} = 1 \ \text{при} \ i = j) \quad (2.5)$$

have negative real parts and thus T-periodic solution always exists for the coupling coordinates in the generating approximation of the type

$$u_j^{\circ} = \sum_{r=1}^{m} u_{rj}(t + \alpha_r) \qquad (j = 1, \ldots, k) \quad (2.6)$$

Here $u_{rj}(t)$ designates the T-periodic function which satisfies the linear system

$$u_{rj}^{\cdot}(t) = a_{1j}u_{r1} + \ldots + a_{kj}u_{rk} + U_{rj}[\varphi_1(t), \ldots, \varphi_n(t)] \quad (2.7)$$
$$(j = 1, \ldots, k, \ r = 1, \ldots, m)$$

The equations in variations of the generating system

$$y_{si}^{\cdot} = p_{1s}(t + \alpha_i)\, y_{1i} + \ldots + p_{ns}(t + \alpha_i)\, y_{ni} \quad (2.8)$$

$$z_j^{\cdot} = a_{1j}z_1 + \ldots + a_{kj}z_k + \sum_{r=1}^{m} [q_{rj}^{(1)}(t + \alpha_r)\, y_{1r} + \ldots + q_{rj}^{(n)}(t + \alpha_r)\, y_{nr}]$$
$$(s = 1, \ldots, n; \ i = 1, \ldots m; \ j = 1, \ldots, k)$$

where

$$p_{sq}(t) = \frac{\partial X_s[\varphi_1(t), \ldots, \varphi_n(t)]}{\partial \varphi_q(t)}, \qquad q_{rj}^{(s)}(t) = \frac{\partial U_{rj}[\varphi_1(t), \ldots, \varphi_n(t)]}{\partial \varphi_s(t)}$$

admit m independent T-periodic solutions

$$y_{si}^{(\nu)} = \varphi_s^{\cdot}(t + \alpha_i)\,\delta_{i\nu}, \qquad Z_j^{(\nu)} = u_{\nu j}^{\cdot}(t + \alpha_\nu) \qquad (\nu = 1, \ldots, m-1) \tag{2.9}$$

$$y_{si}^{(m)} = \varphi_s^{\cdot}(t + \alpha_i), \qquad Z_j^{(m)} = \sum_{r=1}^{m} u_{rj}^{\cdot}(t + \alpha_r)$$

The last k equations of the linear system associated with (2.8) are of the form

$$w_j^{\cdot} + a_{j1}w_1 + \ldots + a_{jk}w_k = 0 \qquad (j = 1, \ldots, k) \tag{2.10}$$

and, consequently, admit a unique trivial T-periodic solution

$$w_j = 0$$

Therefore, the T-periodic solutions of the conjugate system, corresponding to the coordinates of the objects, may be sought from the system

$$v_{si}^{\cdot} + p_{s1}(t + \alpha_i)\,v_{1i} + \ldots + p_{sn}(t + \alpha_i)\,v_{ni} = 0 \tag{2.11}$$

Let us designate by $\eta_s(t)$ the unique T-periodic solution of the system

$$\eta_s^{\cdot} + p_{s1}(t)\,\eta_1 + \ldots + p_{sn}(t)\,\eta_n = 0 \tag{2.12}$$

which satisfies the condition

$$\varphi_1^{\cdot}(t)\,\eta_1(t) + \ldots + \varphi_n^{\cdot}(t)\,\eta_n(t) = 1 \tag{2.13}$$

Then it is obvious that the system (2.11) also admits m linearly independent T-periodic solutions of the form

$$v_{si}^{\cdot(q)} = \eta_1(t + \alpha_i)\,\delta_{iq} \qquad (q = 1, \ldots, m) \tag{2.14}$$

Let us seek the T-periodic solutions of the system (2.11), which satisfy the condition

$$\sum_{i=1}^{m}\sum_{s=1}^{n} y_{si}^{\nu} v_{si}^{(\mu)} = \delta_{\nu\mu} \qquad (\nu, \mu = 1, \ldots, m) \tag{2.15}$$

in form of a linear combination of solutions (2.14)

$$v_{si}^{(\mu)} = \sum_{q=1} C_{q\mu}\eta_s(t + \alpha_i)\,\delta_{iq} = C_{i\mu}\eta_s(t + \alpha_i) \qquad (\mu = 1, \ldots, m) \tag{2.16}$$

The matrix of the coefficients $C_{i\mu}$, which may be found using relations (2.13) and (2.15), will be nondegenerate and is of the form

$$\begin{Vmatrix} 1 & 0 & \ldots & 0 & 0 \\ 0 & 1 & \ldots & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & 1 & 0 \\ -1 & -1 & \ldots & -1 & 1 \end{Vmatrix} \tag{2.17}$$

The equations for the determination of the unknown phases $\alpha_1, \ldots, \alpha_{m-1}$, as it follows from (1.5), will be

$$P_\mu(\alpha_1, \ldots, \alpha_{m-1}) = \sum_{i=1}^{m} \sum_{s=1}^{n} \int_0^T \Big\{ f_{si}\,[\varphi_1(t+\alpha_i), \ldots, \varphi_n(t+\alpha_i)] + \sum_{j=1}^{k} V_{si}^{(j)}\,[\varphi_1(t+\alpha_i), \ldots, \varphi_n(t+\alpha_i)]\, u_j^{\circ} \Big\}\, v_{si}^{(\mu)}\, dt = 0 \tag{2.18}$$

$$(\mu = 1, \ldots, m-1)$$

Finally, after simple transformations taking into account (2.6), (2.16) and (2.17), we arrive at the fundamental system of the problem of internal self-synchronization of almost-like dynamic objects in the presence of weak linear couplings

$$P_\mu(\alpha_1, \ldots, \alpha_{m-1}) = \sum_{s=1}^{n} \int_0^T [(f_{s\mu}(\varphi_1, \ldots, \varphi_n) - f_{sm}(\varphi_1, \ldots, \varphi_n)] + \sum_{j=1}^{k} \sum_{r=1}^{m} [V_{s\mu}^{(j)}(\varphi_1, \ldots, \varphi_n)\, u_{rj}(t+\alpha_r-\alpha_\mu) - V_{sm}^{(j)}(\varphi_1, \ldots, \varphi_n)\, u_{rj}(t+\alpha_r)]\, \eta_s(t)\, dt = 0 \tag{2.19}$$

$$(\mu = 1, \ldots, m-1)$$

The existence of real solutions of system (2.19) with respect to phases $\alpha_1, \ldots, \alpha_{m-1}$ will represent at the same time the condition of the existence of a synchronous regime. Thereby the first approximation to a true period of synchronous motions according to (1.8) will be determined by Formula

$$\alpha^* = -\sum_{s=1}^{n} \int_0^T \Big(f_{sm}(\varphi_1, \ldots, \varphi_n) + \sum_{j=1}^{k} \sum_{r=1}^{m} V_{sn}^{(j)}(\varphi_1, \ldots, \varphi_n)\, u_{rj}(t+\alpha_r) \Big)\, \eta_s(t)\, dt \tag{2.20}$$

The stability of phasing of synchronous motions may be judged by the signs of the real parts of the roots of Equations (2.12)

BIBLIOGRAPHY

1. Blekhman, I.I., Problema sinkhronizatsii dinamicheskikh sistem (The problem of synchronization of dynamic systems). *PMM* Vol.28, № 2, 1964.
2. Malkin, I.G., Nekotorye zadachi teorii nelineinykh kolebanii (Some problems of the theory of nonlinear oscillations). Gostekhteoretizdat,1956.